HEAT TRACE ASYMPTOTICS OF A TIME DEPENDENT PROCESS

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Abstract. We study the heat trace asymptotics defined by a time dependent family of operators of Laplace type which naturally appears for time dependent metrics.

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§1 Introduction

Let M be an m dimensional compact Riemannian manifold with smooth boundary, let V be a smooth vector bundle over M, and let $D : C^\infty(V) \to C^\infty(V)$ be an operator of Laplace type whose coefficients are independent of the parameter t; such an operator is said to be static. There is a canonical connection ∇ on V and a canonical endomorphism E of V so

$$D = -\{\operatorname{Tr}(\nabla^2) + E\}. \tag{1.1.a}$$

Let $x = (x_1, ..., x_m)$ be a system of local coordinates on M. We adopt the Einstein convention and sum over repeated indices. Fix a local frame for V and expand:

$$ds^2_M = g_{\mu\nu} dx^\mu \circ dx^\nu \text{ and } D = -(g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B)$$

where A and B are local sections of $TM \otimes \operatorname{End}(V)$ and $\operatorname{End}(V)$. Let I_V be the identity map on V. The connection 1 form ω of ∇ and the endomorphism E appearing in equation (1.1.a) are given by

$$\begin{aligned}&\omega_\delta = \tfrac{1}{2} g_{\nu\delta}(A^\nu + g^{\mu\sigma}\Gamma_{\mu\sigma}{}^\nu I_V) \text{ and}\\ &E = B - g^{\nu\mu}(\partial_\nu\omega_\mu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma_{\nu\mu}{}^\sigma);\end{aligned} \tag{1.1.b}$$

see [4] for details. Let ';' denote multiple covariant differentiation; we use the Levi-Civita connection on M and the connection of equation (1.1.b) determined by D to differentiate tensors of all types. If $\mathcal{D}$ is a time dependent family of operators of Laplace type, then we expand $\mathcal{D}$ in a Taylor series expansion in t to write $\mathcal{D}$ invariantly in the form:

$$\mathcal{D}u := Du + \textstyle\sum_{r>0} t^r \{\mathcal{G}_{r,}{}^{ij} u_{;ij} + \mathcal{F}_{r,}{}^i u_{;i} + \mathcal{E}_r u\}. \tag{1.1.c}$$

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This setting appears most naturally when defining an adiabatic vacuum in quantum field theory in curved spacetime [1]. If the spacetime is slowly varying, then the time dependent metric describing the cosmological evolution can be expanded in a Taylor series with respect to t. The index r in this situation is then related to the adiabatic order.

Near the boundary, let indices $a, b, ...$ range from 1 through $m-1$ and index a local orthonormal frame for the boundary; let e_m denote the inward unit normal. We assume given a decomposition of the boundary $\partial M = C_{\mathcal{N}} \dot{\sqcup} C_{\mathcal{D}}$ as the disjoint union of closed sets - we permit $C_{\mathcal{N}}$ or $C_{\mathcal{D}}$ to be empty. Let

$$\text{(1.1.d)} \qquad \mathcal{B}u := u|_{C_{\mathcal{D}}} \oplus (u_{;m} + Su + t(T^a u_{;a} + S_1 u))|_{C_{\mathcal{N}}}$$

define the boundary conditions; we can treat both Robin and Dirichlet boundary conditions with this formalism. In the following we shall let $\mathcal{B}_0$ be the static (i.e. time independent) part of the boundary condition; $\mathcal{B}_0 u := u|_{C_{\mathcal{D}}} \oplus (u_{;m} + Su)|_{C_{\mathcal{N}}}$. The reason for including a time-dependence in the boundary condition comes e.g. from considerations of the dynamical Casimir effect; it takes the form given in (1.1.d) for slowly moving boundaries. Here we included only linear powers of t because higher orders do not enter into the asymptotic terms we are going to calculate. Note that by multiplying $\mathcal{B}$ by $(1+T^m)^{-1}$, we can take $T^m = 0$.

If ϕ is the initial temperature distribution, the subsequent temperature distribution $u_\phi(t,x)$ is determined by the equations:

$$\text{(1.1.e)} \qquad (\partial_t + \mathcal{D})u_\phi(t,x) = 0, \ \mathcal{B}u = 0, \ \text{and } u_\phi(0,x) = \phi.$$

Let $\mathcal{K} : \phi \to u_\phi$ be the fundamental solution of the heat equation. If $\mathcal{D}$ and $\mathcal{B}$ are static, then $\mathcal{K} = e^{-tD_{\mathcal{B}}}$. Let ν_M be the Riemannian measure on M. There exists a smooth endomorphism valued kernel $K(t,x,\bar{x},\mathcal{D},\mathcal{B}) : V_{\bar{x}} \to V_x$ so

$$u_\phi(t,x) = (\mathcal{K}\phi)(t,x) = \textstyle\int_M K(t,x,\bar{x},\mathcal{D},\mathcal{B})\phi(\bar{x})d\bar{\nu}_M.$$

For fixed t, the operator $\mathcal{K}(t) : \phi \to \phi(t,\cdot)$ is of trace class. We let

$$\text{(1.1.f)} \qquad a(f,\mathcal{D},\mathcal{B})(t) := \mathrm{Tr}_{L^2}(f\mathcal{K}(t)) = \textstyle\int_M f(x)\,\mathrm{Tr}_{V_x}(K(t,x,x,\mathcal{D},\mathcal{B}))d\nu_M.$$

The function $f \in C^\infty(M)$ is introduced as a localizing or smearing function. As $t \downarrow 0$, one can extend the analysis of [6] from the static setting to show that there is a complete asymptotic expansion of the form

$$\text{(1.1.g)} \qquad a(f,\mathcal{D},\mathcal{B})(t) \sim \textstyle\sum_{n\ge 0} a_n(f,\mathcal{D},\mathcal{B})t^{(n-m)/2}.$$

The asymptotic coefficients $a_n(f,\mathcal{D},\mathcal{B})$ form the focus of our study. We may decompose a_n into an interior and a boundary contribution:

$$a_n(f,\mathcal{D},\mathcal{B}) = a_n^M(f,\mathcal{D}) + a_n^{\partial M}(f,\mathcal{D},\mathcal{B}).$$

The interior invariants vanish if n is odd and do not depend on the boundary condition; the boundary invariants are generically non-zero for all n. Let $N^\mu(f)$

denote the μ^{th} covariant derivative of the smearing function f with respect to e_m. There exist locally computable invariants $a_n^M(x,\mathcal{D})$ and $a_{n,\mu}^{\partial M}(y,\mathcal{D},\mathcal{B})$ defined for interior points $x\in M$ and boundary points $y\in\partial M$ so that

$$\begin{aligned}&a_n^M(f,\mathcal{D})=\textstyle\int_M f(x)a_n^M(x,\mathcal{D})d\nu_M,\ \text{and}\\ &a_n^{\partial M}(f,\mathcal{D},\mathcal{B})=\textstyle\sum_\mu\int_{\partial M}N^\mu(f)a_{n,\mu}^{\partial M}(y,\mathcal{D},\mathcal{B})d\nu_{\partial M}.\end{aligned}\tag{1.1.h}$$

If $\mathcal{D}$ and $\mathcal{B}$ are static, then these are the heat trace asymptotics which have been studied in many contexts previously; $a(1,D,\mathcal{B})=\operatorname{Tr}_{L^2}e^{-tD_{\mathcal{B}}}$. Let R_{ijkl} be the components of the curvature tensor defined by the Levi-Civita connection and let Ω_{ij} be the components of the curvature endomorphism defined by the auxiliary connection ∇ on V. We do not introduce explicit bundle indices for Ω_{ij} and E. Let L_{aa} be the second fundamental form. Let ':' denote multiple covariant differentiation with respect to the Levi-Civita connection of the boundary and the connection defined by D. We refer to [2] and [4] for the proof of the following result for static D; see also related work [3, 7, 8, 9].

1.2 Theorem.

(1) $a_0^M(f,D)=(4\pi)^{-m/2}\int_M f\operatorname{Tr}(I_V)d\nu_M$.

(2) $a_2^M(f,D)=(4\pi)^{-m/2}\frac16\int_M f\operatorname{Tr}(R_{ijji}I_V+6E)d\nu_M$.

(3) $a_4^M(f,D)=(4\pi)^{-m/2}\frac1{360}\int_M f\operatorname{Tr}\{60E_{;kk}+60R_{ijji}E+180E^2+30\Omega_{ij}\Omega_{ij}$
$+(12R_{ijji;kk}+5R_{ijji}R_{kllk}-2R_{ijki}R_{ljkl}+2R_{ijkl}R_{ijkl})I_V\}d\nu_M$.

(4) $a_0^{\partial M}(f,D,\mathcal{B})=0$.

(5) $a_1^{\partial M}(f,D,\mathcal{B})=-(4\pi)^{(1-m)/2}\frac14\int_{C_D}f\operatorname{Tr}(I_V)d\nu_{\partial M}$
$+(4\pi)^{(1-m)/2}\frac14\int_{C_N}f\operatorname{Tr}(I_V)d\nu_{\partial M}$.

(6) $a_2^{\partial M}(f,D,\mathcal{B})=(4\pi)^{-m/2}\frac16\int_{C_D}\operatorname{Tr}\{2fL_{aa}I_V-3f_{;m}I_V\}d\nu_{\partial M}$
$+(4\pi)^{-m/2}\frac16\int_{C_N}\operatorname{Tr}\{f(2L_{aa}I_V+12S)+3f_{;m}I_V\}d\nu_{\partial M}$.

(7) $a_3^{\partial M}(f,D,\mathcal{B})=-(4\pi)^{(1-m)/2}\frac1{384}\int_{C_D}\operatorname{Tr}\{96fE+f(16R_{ijji}$
$-8R_{amma}+7L_{aa}L_{bb}-10L_{ab}L_{ab})I_V-30f_{;m}L_{aa}I_V+24f_{;mm}I_V\}d\nu_{\partial M}$
$+(4\pi)^{(1-m)/2}\frac1{384}\int_{C_N}\operatorname{Tr}(96fE+f(16R_{ijji}-8R_{amma}+13L_{aa}L_{bb}$
$+2L_{ab}L_{ab})I_V+f(96SL_{aa}+192S^2)+f_{;m}(6L_{aa}I_V+96S)$
$+24f_{;mm}I_V\}d\nu_{\partial M}$.

(8) $a_4^{\partial M}(f,D,\mathcal{B})=(4\pi)^{-m/2}\frac1{360}\int_{C_D}\operatorname{Tr}\{f(-120E_{;m}+120EL_{aa})$
$+f(-18R_{ijji;m}+20R_{ijji}L_{aa}+4R_{amam}L_{bb}-12R_{ambm}L_{ab}+4R_{abcb}L_{ac}$
$+24L_{aa:bb}+\frac{40}{21}L_{aa}L_{bb}L_{cc}-\frac{88}{7}L_{ab}L_{ab}L_{cc}+\frac{320}{21}L_{ab}L_{bc}L_{ac})I_V-180f_{;m}E$
$+f_{;m}(-30R_{ijji}-\frac{180}{7}L_{aa}L_{bb}+\frac{60}{7}L_{ab}L_{ab})I_V+24f_{;mm}L_{aa}I_V$
$-30f_{;iim}I_V\}d\nu_{\partial M}$
$+(4\pi)^{-m/2}\frac1{360}\int_{C_N}\operatorname{Tr}\{f(240E_{;m}+120EL_{aa})+f(42R_{ijji;m}+24L_{aa:bb}$
$+20R_{ijji}L_{aa}+4R_{amam}L_{bb}-12R_{ambm}L_{ab}+4R_{abcb}L_{ac}+\frac{40}{3}L_{aa}L_{bb}L_{cc}$
$+8L_{ab}L_{ab}L_{cc}+\frac{32}{3}L_{ab}L_{bc}L_{ac})I_V+f(720SE+120SR_{ijji}+144SL_{aa}L_{bb}$
$+48SL_{ab}L_{ab}+480S^2L_{aa}+480S^3+120S_{:aa})+f_{;m}(180E+72SL_{aa}$
$+240S^2)+f_{;m}(30R_{ijji}+12L_{aa}L_{bb}+12L_{ab}L_{ab})I_V+120f_{;mm}S$
$+24f_{;mm}L_{aa}I_V+30f_{;iim}I_V\}d\nu_{\partial M}$.

The main result of this paper is the following result which extends Theorem 1.2 to the time dependent setting:

1.3 Theorem.

(1) $a_0^M(f,\mathcal{D}) = a_0^M(f,D)$.

(2) $a_2^M(f,\mathcal{D}) = a_2^M(f,D) + (4\pi)^{-m/2}\frac{1}{6}\int_M f\,\mathrm{Tr}(\frac{3}{2}\mathcal{G}_{1,ii})d\nu_M$.

(3) $a_4^M(f,\mathcal{D}) = a_4^M(f,D) + (4\pi)^{-m/2}\frac{1}{360}\int_M f\,\mathrm{Tr}(\frac{45}{4}\mathcal{G}_{1,ii}\mathcal{G}_{1,jj} + \frac{45}{2}\mathcal{G}_{1,ij}\mathcal{G}_{1,ij}$
$+60\mathcal{G}_{2,ii} - 180\mathcal{E}_1 + 15\mathcal{G}_{1,ii}R_{jkkj} - 30\mathcal{G}_{1,ij}R_{ikkj} + 90\mathcal{G}_{1,ii}E + 60\mathcal{F}_{1,i;i}$
$15\mathcal{G}_{1,ii;jj} - 30\mathcal{G}_{1,ij;ij})d\nu_M$.

(4) $a_n^{\partial M}(f,\mathcal{D},\mathcal{B}) = a_n^{\partial M}(f,D,\mathcal{B}_0)$ *for* $n \le 2$.

(5) $a_3^{\partial M}(f,\mathcal{D},\mathcal{B}) = a_3^{\partial M}(f,D,\mathcal{B}_0) + (4\pi)^{(1-m)/2}\frac{1}{384}\int_{C_D} f\,\mathrm{Tr}(-24\mathcal{G}_{1,aa})d\nu_{\partial M}$
$+(4\pi)^{(1-m)/2}\frac{1}{384}\int_{C_N} f\,\mathrm{Tr}(24\mathcal{G}_{1,aa})d\nu_{\partial M}$.

(6) $a_4^{\partial M}(f,\mathcal{D},\mathcal{B}) = a_4^{\partial M}(f,D,\mathcal{B}_0) + (4\pi)^{-m/2}\frac{1}{360}\int_{C_D}\mathrm{Tr}\{f(30\mathcal{G}_{1,aa}L_{bb}$
$-60\mathcal{G}_{1,mm}L_{bb} + 30\mathcal{G}_{1,ab}L_{ab} + 30\mathcal{G}_{1,mm;m} - 30\mathcal{G}_{1,aa;m} + 0\mathcal{G}_{1,am;a} - 30\mathcal{F}_{1,m})$
$+f_{;m}(-45\mathcal{G}_{1,aa} + 45\mathcal{G}_{1,mm})\}d\nu_{\partial M}$
$+(4\pi)^{-m/2}\frac{1}{360}\int_{C_N}\mathrm{Tr}\{f(30\mathcal{G}_{1,aa}L_{bb} + 120\mathcal{G}_{1,mm}L_{bb} - 150\mathcal{G}_{1,ab}L_{ab}$
$-60\mathcal{G}_{1,mm;m} + 60\mathcal{G}_{1,aa;m} + 0\mathcal{G}_{1,am;a} + 150\mathcal{F}_{1,m} + 180S\mathcal{G}_{1,aa}$
$-180S\mathcal{G}_{1,mm} + 360S_1 + 0T_{a:a}) + f_{;m}(45\mathcal{G}_{1,aa} - 45\mathcal{G}_{1,mm})\}d\nu_{\partial M}$.

Here is a brief outline to this paper. In §2, we use invariance theory and dimensional analysis to study the general form of the invariants $a_n(f,\mathcal{D},\mathcal{B})$. We shall use $\mathcal{B}^-$ for Dirichlet and $\mathcal{B}^+$ for Robin boundary conditions. We shall show, for example, that there exist constants c_0 and $e_1^\pm$ so that:

$$\begin{aligned}
&a_2^M(f,\mathcal{D}) = a_2^M(f,D) + (4\pi)^{-m/2}\tfrac{1}{6}\textstyle\int_M f\,\mathrm{Tr}(c_0\mathcal{G}_{1,ii})d\nu_M \text{ and}\\
&a_3^{\partial M}(f,\mathcal{D},\mathcal{B}) = a_3^{\partial M}(f,D,\mathcal{B}_0) + (4\pi)^{-(m-1)/2}\tfrac{1}{384}\textstyle\int_{C_D} f\,\mathrm{Tr}(e_1^-\mathcal{G}_{1,aa})d\nu_{\partial M}\\
&\qquad + (4\pi)^{-(m-1)/2}\tfrac{1}{384}\textstyle\int_{C_N} f\,\mathrm{Tr}(e_1^+\mathcal{G}_{1,aa})d\nu_{\partial M};
\end{aligned}$$

we refer to Lemma 2.1 for further details. The interior invariants will be described by constants $\{c_i\}_{i=0}^{10}$, the boundary invariants for Neumann boundary conditions will be described by constants $\{e_i^+\}_{i=1}^{15}$, and the boundary invariants for Dirichlet boundary conditions will be described by constants $\{e_i^-\}_{i=1}^{11}$. We use the localizing function f to decouple the interior and the boundary integrals; with the exception of Lemma 2.4, there is no interaction between the unknown constants $\{c_i\}$, $\{e_j^-\}$, and $\{e_k^+\}$. A priori, those constants could depend on the dimension. In Lemma 2.3, we will use product formulas to dimension shift and show the constants are dimension free. We complete the proof of Theorem 1.3 by evaluating these unknown constants; the values we shall derive are summarized in Table 2.2.

We use various functorial properties to derive relations among these constants. For example, in Lemma 2.4, we use the product formulas of Lemma 2.3 to show that $c_5 = 10c_0$. The functorial properties that these time dependent invariants satisfy and which are discussed in §3-§6 are new and have not been used previously in other calculations of the heat trace asymptotics. Thus we believe they are of interest in their own right. It is one of the features of the functorial method that one has to work in great generality even if one is only interested in special cases.

We found it necessary, for example, to consider the very general time dependent boundary conditions of equation (1.1.d) to ensure that the class of boundary conditions was invariant under the gauge and coordinate transformations employed in §4 and §5. We work with scalar operators as the (possible) non-commutativity of the endomorphisms in the vector valued case plays no role in the evaluation of a_n for $n \le 4$.

We summarize the five functorial properties we shall use as follows. In §2, we consider a product manifold $M = M_1 \times M_2$ where ∂M_2 is empty, and an operator of the form $\mathcal{D} = \mathcal{D}_1 \otimes 1 + 1 \otimes \mathcal{D}_2$. In Lemma 2.4, we show that

$$a_n(f_1 f_2, \mathcal{D}, \mathcal{B}) = \textstyle\sum_{p+q=n} a_p(f_1, \mathcal{D}_1, \mathcal{B}) a_q(f_2, \mathcal{D}_2).$$

In §3, we rescale the time parameter t. Let D and $\mathcal{B}$ be static operators. Let $\mathcal{D} := (1 + 2\alpha t + 3\beta t^2) D$. In Lemma 3.1, we show that:

$$\begin{aligned}
&a_2(f, \mathcal{D}, \mathcal{B}) = a_2(f, D, \mathcal{B}) - \tfrac{m}{2}\alpha a_0(f, D, \mathcal{B})\\
&a_3(f, \mathcal{D}, \mathcal{B}) = a_3(f, D, \mathcal{B}) - \tfrac{m-1}{2}\alpha a_1(f, D, \mathcal{B})\\
&a_4(f, \mathcal{D}, \mathcal{B}) = a_4(f, D, \mathcal{B}) - \tfrac{m-2}{2}\alpha a_2(f, D, \mathcal{B}) + (\tfrac{m(m+2)}{8}\alpha^2 - \tfrac{m}{2}\beta) a_0(f, D, \mathcal{B}).
\end{aligned}$$

In §4, we make a time dependent gauge transformation. We assume D and $\mathcal{B}$ are static. Let $\mathcal{D}_\varrho := e^{-t\varrho\Psi} D e^{t\varrho\Psi} + \varrho\Psi$. We also gauge transform the boundary condition $\mathcal{B}$ to define $\mathcal{B}_\varrho$. In Lemma 4.1, we show that :

$$\tfrac{\partial}{\partial\varrho}\{a_n(f, \mathcal{D}_\varrho, \mathcal{B}_\varrho)\}|_{\varrho=0} = -a_{n-2}(f\Psi, D, \mathcal{B}).$$

In §5, we make a time dependent coordinate transformation. Let Δ be the scalar Laplacian and let $\mathcal{B}$ be static. Let $\Phi_\varrho : (t, x_1, x_2) \to (t, x_1 + t\varrho\Xi, x_2)$ where ϱ is an auxiliary parameter. We set $\mathcal{D}_\varrho := \Phi_\varrho^*(\partial_t + \Delta) - \partial_t$ and $\mathcal{B}_\varrho := \Phi_\varrho^*(\mathcal{B})$. Let $d\nu_M := g dx^1 dx^2$. In Lemma 5.1, we show that:

$$\tfrac{\partial}{\partial\varrho}\{a_n(f, \mathcal{D}_\varrho, \mathcal{B}_\varrho)\}|_{\varrho=0} = -\tfrac{1}{2} a_{n-2}(g^{-1}\partial_1(gf\Xi), \Delta, \mathcal{B}).$$

In §6, we assume given a second order operator Q which commutes with a static operator D of Laplace type. We define $D_\varrho := D + \varrho Q$ and define a suitable boundary condition $\mathcal{B}_\varrho$. We also define $\mathcal{D}_\varrho := D + 2t\varrho Q$ and show

$$\tfrac{\partial}{\partial\varrho}\{a_n(f, \mathcal{D}_\varrho, \mathcal{B})\}|_{\varrho=0} = \tfrac{\partial}{\partial\varrho}\{a_{n-2}(f, D_\varrho, \mathcal{B}_\varrho)\}|_{\varrho=0}.$$

In each section, we use the relevant functorial properties to derive relations among the unknown coefficients; these relations are contained in Lemmas 2.4, 3.2, 4.2, and 5.2. These relations suffice to determine the unknown coefficients and thereby complete the proof of Theorem 1.3. As the computations are somewhat long and technical, we have derived more equations than are needed as a consistency check; this is typical in such computations.

§2 Invariance Theory, dimensional analysis, and dimension shifting

We begin the proof of Theorem 1.3 by establishing the general form of the invariants a_n^M and $a_n^{\partial M}$ for $n \le 4$. Let $(D, \mathcal{B}_0)$ be the static operator and boundary condition determined by $(\mathcal{D}, \mathcal{B})$.

2.1 Lemma. *There exist constants so that*

(1) $a_0^M(f, \mathcal{D}) = a_0^M(f, D)$ *and* $a_i^{\partial M}(f, \mathcal{D}, \mathcal{B}) = a_i^{\partial M}(f, D, \mathcal{B}_0)$ *for* $i \le 2$.

(2) $a_2^M(f, \mathcal{D}) = a_2^M(f, D) + (4\pi)^{-m/2}\frac{1}{6}\int_M f\,\mathrm{Tr}\{c_0\mathcal{G}_{1,ii}\}d\nu_M$.

(3) $a_4^M(f, \mathcal{D}) = a_4^M(f, D) + (4\pi)^{-m/2}\frac{1}{360}\int_M f\,\mathrm{Tr}\{c_1\mathcal{G}_{1,ii}\mathcal{G}_{1,jj} + c_2\mathcal{G}_{1,ij}\mathcal{G}_{1,ij}$
$+c_3\mathcal{G}_{2,ii} + c_4\mathcal{E}_1 + c_5\mathcal{G}_{1,ii}R_{jkkj} + c_6\mathcal{G}_{1,ij}R_{ikkj} + c_7\mathcal{G}_{1,ii}E + c_8\mathcal{F}_{1,i;i}$
$+c_9\mathcal{G}_{1,ii;jj} + c_{10}\mathcal{G}_{1,ij;ij}\}d\nu_M$.

(4) $a_3^{\partial M}(f, \mathcal{D}, \mathcal{B}) = a_3^{\partial M}(f, D, \mathcal{B}_0) + (4\pi)^{(1-m)/2}\frac{1}{384}\int_{C_D} f\,\mathrm{Tr}(e_1^-\mathcal{G}_{1,aa}$
$+e_2^-\mathcal{G}_{1,mm})d\nu_{\partial M} + (4\pi)^{(1-m)/2}\frac{1}{384}\int_{C_N} f\,\mathrm{Tr}(e_1^+\mathcal{G}_{1,aa} + e_2^+\mathcal{G}_{1,mm})d\nu_{\partial M}$.

(5) $a_4^{\partial M}(f, \mathcal{D}, \mathcal{B}) = a_4(f, D, \mathcal{B}_0) + (4\pi)^{-m/2}\frac{1}{360}\int_{C_D}\mathrm{Tr}\{f(e_3^-\mathcal{G}_{1,aa}L_{bb}$
$+e_4^-\mathcal{G}_{1,mm}L_{bb} + e_5^-\mathcal{G}_{1,ab}L_{ab} + e_6^-\mathcal{G}_{1,mm;m} + e_7^-\mathcal{G}_{1,aa;m}$
$+e_8^-\mathcal{G}_{1,am;a} + e_9^-\mathcal{F}_{1,m}) + f_{;m}(e_{10}^-\mathcal{G}_{1,aa} + e_{11}^-\mathcal{G}_{1,mm})\}d\nu_{\partial M}$
$+(4\pi)^{-m/2}\frac{1}{360}\int_{C_N}\mathrm{Tr}\{f(e_3^+\mathcal{G}_{1,aa}L_{bb} + e_4^+\mathcal{G}_{1,mm}L_{bb} + e_5^+\mathcal{G}_{1,ab}L_{ab}$
$+e_6^+\mathcal{G}_{1,mm;m} + e_7^+\mathcal{G}_{1,aa;m} + e_8^+\mathcal{G}_{1,am;a} + e_9^+\mathcal{F}_{1,m} + e_{12}^+S\mathcal{G}_{1,aa}$
$+e_{13}^+S\mathcal{G}_{1,mm} + e_{14}^+S_1 + e_{15}^+T_{a:a}) + f_{;m}(e_{10}^+\mathcal{G}_{1,aa} + e_{11}^+\mathcal{G}_{1,mm})\}d\nu_{\partial M}$.

Proof. We use dimensional analysis - this involves studying the behavior of these invariants under rescaling and is described in [4] in the static setting. We assign weight 2 to R, Ω, E and T_a and weight 3 to S_1. We assign weight 1 to S and L_{ab}. We increase the weight by 1 for each explicit covariant derivative which appears. Thus, for example, the terms $E_{;kk}$, $\Omega_{ij}\Omega_{ij}$, and $R_{ijkl}R_{ijkl}$ are all of degree 4. The integrands appearing in a_n^M and $a_n^{\partial M}$ are weighted homogeneous of degree n and $n-1$. The structure groups are $O(m)$ and $O(m-1)$ respectively. H. Weyl's Theorem [10] shows that all orthogonal invariants are given by contractions of indices. The assertions of the Lemma now follow by writing down a spanning set for the space of invariants. We remark that since $\mathcal{G}_{1,ij} = \mathcal{G}_{1,ji}$, the invariant $\mathcal{G}_{1,ij}\Omega_{ij}$ does not appear. □

We will complete the proof of Theorem 1.3 by evaluating the unknown coefficients of Lemma 2.1. The remainder of this paper is devoted to deriving the values in the following table:

Table 2.2

$c_0 = \frac{3}{2}$	$c_1 = \frac{45}{4}$	$c_2 = \frac{45}{2}$	$c_3 = 60$	$c_4 = -180$	$c_5 = 15$
$c_6 = -30$	$c_7 = 90$	$c_8 = 60$	$c_9 = 15$	$c_{10} = -30$	
$e_1^- = -24$	$e_2^- = 0$	$e_3^- = 30$	$e_4^- = -60$	$e_5^- = 30$	$e_6^- = 30$
$e_7^- = -30$	$e_8^- = 0$	$e_9^- = -30$	$e_{10}^- = -45$	$e_{11}^- = 45$	
$e_1^+ = 24$	$e_2^+ = 0$	$e_3^+ = 30$	$e_4^+ = 120$	$e_5^+ = -150$	$e_6^+ = -60$
$e_7^+ = 60$	$e_8^+ = 0$	$e_9^+ = 150$	$e_{10}^+ = 45$	$e_{11}^+ = -45$	$e_{12}^+ = 180$
$e_{13}^+ = -180$	$e_{14}^+ = 360$	$e_{15}^+ = 0$			

The (possible) non-commutativity of the endomorphisms in the vector valued case plays no role in the invariants of Lemma 2.1. We therefore suppose V to be the trivial bundle hence forth and omit the trace from our formulas to simplify the notation as we will be dealing with scalar operators on $C^\infty(M)$. We also set $e_i^- = 0$ for $i \geq 12$ to have a common formalism; these constants describe invariants which involve S, S_1, and T_a and which are therefore not relevant for Dirichlet boundary conditions.
A-priori, the constants c_i and $e_i^\pm$ might depend upon the dimension. Fortunately, this turns out not to be the case; the dependence upon the dimension is contained in the multiplicative normalizing factors of $(4\pi)^*$. Let $\mathcal{D}_i$ be smooth time dependent families of operators of Laplace type over manifolds M_i for $i = 1, 2$. We suppose M_2 is closed. Let $M := M_1 \times M_2$, let $\mathcal{D} := \mathcal{D}_1 + \mathcal{D}_2$, and let the boundary condition for M be induced from the corresponding boundary condition for M_1.

2.3 Lemma. *Adopt the notation established above.*

(1) $a_n^M(f_1f_2, \mathcal{D}) = \sum_{p+q=n} a_p^{M_1}(f_1, \mathcal{D}_1)a_q^{M_2}(f_2, \mathcal{D}_2)$
(2) $a_n^{\partial M}(f_1f_2, \mathcal{D}, \mathcal{B}) = \sum_{p+q=n} a_p^{\partial M_1}(f_1, \mathcal{D}_1, \mathcal{B})a_q^{M_2}(f_2, \mathcal{D}_2)$.
(3) *The constants of Lemma 2.1 do not depend upon the dimension m.*

Proof. We use equation (1.1.e) to check that $u_{\phi_1\cdot\phi_2} = u_{\phi_1} \cdot u_{\phi_2}$. This shows the kernel function on M is the product of the corresponding kernel functions on M_1 and on M_2; assertions (1) and (2) now follow. Let $(M, \mathcal{D}_M, \mathcal{B})$ be given. Let S^1 be the unit circle with the usual flat metric and usual periodic parameter θ. Let $D_S = -\partial_\theta^2$ on the trivial line bundle. Let $\mathcal{D}_{M\times S^1} = \mathcal{D}_M + D_S$. Then $a_p(\theta, D_S) = 0$ for $p > 0$ and $a_0(\theta, D_S) = (4\pi)^{-1/2}$; see [4] for details. Thus $p = n$ and $q = 0$ in assertions (1) and (2) so $a_n(f_1, \mathcal{D}_{M\times S^1}) = (4\pi)^{-1/2}a_n(f_1, \mathcal{D}_M, \mathcal{B})$. It now follows that $c_i(m+1) = c_i(m)$ and $e_i^\pm(m+1) = e_i^\pm(m)$. □

We use the product formulas of Lemma 2.3 to prove the following Lemma:

2.4 Lemma. *We have $c_1 = 5c_0^2$, $c_5 = 10c_0$, $c_7 = 60c_0$, $e_1^- = -16c_0$, $e_3^- = 20c_0$, $e_{10}^- = -30c_0$, $e_1^+ = 16c_0$, $e_3^+ = 20c_0$, $e_{10}^+ = 30c_0$, and $e_{12}^+ = 120c_0$.*

Proof. We apply Lemma 2.3 and study the cross terms arising in $a_{p+q}(f_1f_2, \mathcal{D}, \mathcal{B})$ from $a_p(f_1, \mathcal{D}_1, \mathcal{B}_1)a_q(f_2, \mathcal{D}_2)$. We let indices r and s index M_1 and indices u and v index M_2. We use Theorem 1.2 and equate coefficients of suitable expressions to derive the following systems of equations from which the Lemma will follow:

$2c_1 = 360(\frac{1}{6}c_0)(\frac{1}{6}c_0)$ $[f_1f_2\mathcal{G}_{1,rr}\mathcal{G}_{1,uu}]$	$c_5 = 360(\frac{1}{6})(\frac{1}{6}c_0)$ $[f_1f_2R_{rssr}\mathcal{G}_{1,uu}]$
$c_7 = 360(\frac{1}{6}c_0)$ $[f_1f_2E_1\mathcal{G}_{1,uu}]$	$e_1^\pm = 384(\pm\frac{1}{4})(\frac{1}{6}c_0)$ $[f_1f_2\mathcal{G}_{1,uu}]$
$e_3^\pm = 360(\frac{1}{3})(\frac{1}{6}c_0)$ $[f_1f_2L_{rr}\mathcal{G}_{1,uu}]$	$e_{10}^\pm = 360(\pm\frac{1}{2})(\frac{1}{6}c_0)$ $[f_{1;m}f_2\mathcal{G}_{1,uu}]$
$e_{12}^+ = 360(2)(\frac{1}{6}c_0)$ $[fS\mathcal{G}_{1,uu}]$ □	

§3 Rescaling the time parameter

Let D and $\mathcal{B}$ be static. Let $\alpha, \beta \in \mathbb{R}$. We define a time dependent family of operators of Laplace type by setting: $\mathcal{D} := (1 + 2\alpha t + 3\beta t^2)D$.

3.1 Lemma.

(1) $a_2(f,\mathcal{D},\mathcal{B}) = a_2(f,D,\mathcal{B}) - \frac{m}{2}\alpha a_0(f,D,\mathcal{B})$.

(2) $a_3(f,\mathcal{D},\mathcal{B}) = a_3(f,D,\mathcal{B}) - \frac{m-1}{2}\alpha a_1(f,D,\mathcal{B})$.

(3) $a_4(f,\mathcal{D},\mathcal{B}) = a_4(f,D,\mathcal{B}) - \frac{m-2}{2}\alpha a_2(f,D,\mathcal{B}) + (\frac{m(m+2)}{8}\alpha^2 - \frac{m}{2}\beta)a_0(f,D,\mathcal{B})$.

Proof. Let $u_0 = e^{-tD_\mathcal{B}}\phi$ and let $u(t,x) := u_0(t+\alpha t^2+\beta t^3, x)$. Then:

$$\mathcal{D}u(t,x) = (1+2\alpha t+3\beta t^2)(Du_0)(t+\alpha t^2+\beta t^3, x)$$
$$\partial_t u(t,x) = (1+2\alpha t+3\beta t^2)(\partial_t u_0)(t+\alpha t^2+\beta t^3, x).$$

This shows that $(\partial_t + \mathcal{D})u = 0$. Since $u(0,x) = u_0(0,x) = \phi(x)$ and $\mathcal{B}u = 0$, the relations of equation (1.1.e) are satisfied so

$$K(t,x,\bar{x},\mathcal{D},\mathcal{B}) = K(t+\alpha t^2+\beta t^3, x, \bar{x}, D, \mathcal{B}).$$

The Lemma will then follow from the expansions:

$$a(f,\mathcal{D},\mathcal{B})(t) \sim \textstyle\sum_n t^{-m/2}(1+\alpha t+\beta t^2)^{(n-m)/2} a_n(f,D,\mathcal{B})t^{n/2}$$
$$(1+\alpha t+\beta t^2)^j \sim 1 + \alpha j t + (\tfrac{j(j-1)}{2}\alpha^2 + j\beta)t^2 + O(t^3) \quad \square$$

We apply Theorem 1.2 and Lemma 3.1 to derive the following relationships:

3.2 Lemma.

(1) $c_0 = \frac{3}{2}$, $c_1 = \frac{45}{4}$, $c_2 = \frac{45}{2}$, $c_3 = 60$, $c_4 = -180$, $c_5 = 15$, $c_6 = -30$, $c_7 = 90$.

(2) $e_1^\pm = \pm 24$, $e_2^\pm = 0$, $e_3^\pm = 30$, $e_4^\pm + e_5^\pm = -30$, $e_{10}^\pm = \pm 45$, $e_{11}^\pm = \mp 45$.

(3) $e_{12}^+ = 180$, $e_{13}^+ = -180$.

Proof. We have $\mathcal{G}_{1,ij} = -2\alpha g_{ij}$, $\mathcal{F}_{1,i} = 0$, $\mathcal{G}_{2,ij} = -3\beta g_{ij}$, and $\mathcal{E}_1 = -2\alpha E$. Thus $\mathcal{G}_{1,ii;jj} = 0$, $\mathcal{G}_{1,ij;ij} = 0$, and $\mathcal{F}_{1,i;i} = 0$. We equate coefficients of suitable expressions in Lemma 3.1 to derive the following systems of equations from which the Lemma will follow. Note that since m is arbitrary, equations involving this parameter can give rise to more than one relation.

$-2mc_0 = -6\frac{m}{2}$	$[\alpha f]$ in a_2^M
$4(m^2c_1 + mc_2) = 360\frac{m(m+2)}{8}$	$[\alpha^2 f]$ in a_4^M
$-3mc_3 = -360\frac{m}{2}$	$[\beta f]$ in a_4^M
$-2(c_4 + mc_7) = -360\frac{m-2}{12}6$	$[\alpha f E]$ in a_4^M
$-2(mc_5 + c_6) = -360\frac{m-2}{12}$	$[\alpha f R_{ijji}]$ in a_4^M
$-2\{(m-1)e_1^\pm + e_2^\pm\} = -384(\frac{m-1}{2})(\pm\frac{1}{4})$	$[\alpha f]$ in $a_3^{\partial M}$
$-2\{(m-1)e_3^\pm + e_4^\pm + e_5^\pm\} = -360(\frac{m-2}{2})(\frac{1}{3})$	$[\alpha f L_{aa}]$ in $a_4^{\partial M}$
$-2\{(m-1)e_{10}^\pm + e_{11}^\pm\} = -360(\frac{m-2}{2})(\pm\frac{1}{2})$	$[\alpha f_{;m}]$ in $a_4^{\partial M}$
$-2\{(m-1)e_{12}^+ + e_{13}^+\} = -360(\frac{m-2}{2})(2)$.	$[\alpha f S]$ in $a_4^{\partial M}$ $\square$

§4 Time dependent gauge transformations

Let $\mathcal{D}_\varrho := e^{-t\varrho\Psi} D e^{t\varrho\Psi} + \varrho\Psi$. If $\mathcal{B}u = u_{;m} + Su$ is the Robin boundary operator, we gauge transform the boundary condition to define $\mathcal{B}_\varrho := \nabla_m + S + tS_1$ with $S_1 = \varrho\Psi_{;m}$; the Dirichlet boundary operator is unchanged.

4.1 Lemma. *We have* $\frac{\partial}{\partial\varrho}\{a_n(f, \mathcal{D}_\varrho, \mathcal{B}_\varrho)\}|_{\varrho=0} = -a_{n-2}(f\Psi, D, \mathcal{B})$.

Proof. Let $u_0 := e^{-tD_\mathcal{B}}\phi$ and let $u := e^{-t\varrho\Psi}u_0$. We show u satisfies the relations of (1.1.e) by computing:

$$\begin{aligned}
&\partial_t u(t,x) = e^{-t\varrho\Psi}(\partial_t - \varrho\Psi)u_0,\ \mathcal{D}_\varrho u(t,x) = e^{-t\varrho\Psi}(D + \varrho\Psi)u_0,\\
&(\partial_t + \mathcal{D}_\varrho)u = e^{-t\varrho\Psi}(\partial_t + D)u_0 = 0, \text{ and } u(0,x) = u_0(x) = \phi(x).
\end{aligned}$$

Dirichlet boundary conditions are preserved. With Robin boundary conditions,

$$u_{;m} + Su + tS_1u = e^{-t\varrho\Psi}(u_{0;m} - t\varrho\Psi_{;m}u_0 + Su_0 + t\varrho\Psi_{;m}u_0) = 0.$$

Thus $K(\cdot, \mathcal{D}_\varrho, \mathcal{B}_\varrho) = e^{-t\varrho\Psi}K(\cdot, D, \mathcal{B})$. The Lemma now follows. □

We use Lemma 4.1 to obtain some additional relationships:

4.2 Lemma. *We have* $c_8 = 60$, $e_9^- = -30$, *and* $e_{14}^+ - 2e_9^+ = 60$.

Proof. Let Ψ vanish on ∂M. We apply Lemma 4.1 with $M = [0,1]$ and $D = -\partial_\theta^2$. We work modulo terms which are $O(\varrho^2)$ and compute:

$$\begin{aligned}
&D_\varrho \equiv D + \varrho\Psi - 2t\varrho\Psi_{;\theta}\partial_\theta - t\varrho\Psi_{;\theta\theta},\\
&\mathcal{B}_\varrho^+ \equiv \nabla_m + S + t\varrho\Psi_{;m},\ S_1 \equiv \varrho\Psi_{;\theta},\\
&E \equiv -\varrho\Psi,\ \mathcal{F}_{1,m} \equiv -2\varrho\Psi_{;\theta},\ \mathcal{E}_1 \equiv -\varrho\Psi_{;\theta\theta}.
\end{aligned}$$

We study $\frac{\partial}{\partial\varrho}\{a_4^M\}|_{\varrho=0}$ and $\frac{\partial}{\partial\varrho}\{a_4^{\partial M}\}|_{\varrho=0}$:

$\frac{\partial}{\partial\varrho}\{60E_{;ii}\}\|_{\varrho=0} \equiv -60\Psi_{;\theta\theta}$	$\frac{\partial}{\partial\varrho}\{-180\mathcal{E}_1\}\|_{\varrho=0} \equiv 180\Psi_{;\theta\theta}$
$\frac{\partial}{\partial\varrho}\{c_8\mathcal{F}_{1,i;i}\}\|_{\varrho=0} \equiv -2c_8\Psi_{;\theta\theta}$	$\frac{\partial}{\partial\varrho}\{e_{14}^+S_1\}\|_{\varrho=0} \equiv e_{14}^+\Psi_{;\theta}$
$\frac{\partial}{\partial\varrho}\{(-120^-, 240^+)E_{;m}\}\|_{\varrho=0} \equiv (120^-, -240^+)\Psi_{;\theta}$	$\frac{\partial}{\partial\varrho}\{e_9^\pm\mathcal{F}_{1,m}\}\|_{\varrho=0} \equiv -2e_9^\pm\Psi_{;\theta}$

Here the notation $(-120^-, 240^+)$ indicates that the coefficient for Dirichlet $\mathcal{B}^-$ and Neumann $\mathcal{B}^+$ boundary conditions is -120 and 240. As $-a_2^M(f\Psi, D) = 0$ and $-a_2^{\partial M}(f\Psi, D, \mathcal{B}^\pm) = -\frac{1}{360}(4\pi)^{-1/2}\int_{\partial M} \pm 180(f\Psi)_{;m}$, we use Lemma 4.1 to derive the following equations from which the Lemma will follow:

$$\begin{aligned}
0 &= -60 + 180 - 2c_8,\\
-180 &= -2e_9^+ + e_{14}^+ - 240,\\
180 &= 120 - 2e_9^-. \quad \square
\end{aligned}$$

§5 Time dependent coordinate transformations

In this section, we study time dependent coordinate transformations and make a coordinate transformation that mixes up the spatial and the temporal coordinates. This technique was also used in [5] to study the heat content asymptotics. We work in a very specific context but note the Lemma holds true in much greater generality. Let $M := S^1 \times [0,1]$ with $ds^2 = e^{2\psi_1}dx_1^2 + e^{2\psi_2}dx_2^2$. Let $d\nu_M := gdx^1dx^2$. Let $\Xi \in C^\infty(M)$ have compact support near some point $P \in M$. Let Δ be the scalar Laplacian and let $\mathcal{B}$ be a static boundary condition. Define:

$$\begin{aligned}&\Phi_\varrho(t,x_1,x_2) := (t, x_1 + t\varrho\Xi, x_2),\\ &\mathcal{D}_\varrho := \Phi_\varrho^*(\partial_t + \Delta) - \partial_t, \text{ and } \mathcal{B}_\varrho := \Phi_\varrho^*(\mathcal{B}).\end{aligned}$$

5.1 Lemma. *We have* $\frac{\partial}{\partial\varrho}a_n(f,\mathcal{D}_\varrho,\mathcal{B}_\varrho)|_{\varrho=0} = -\frac{1}{2}a_{n-2}(g^{-1}\partial_1(gf\Xi),\Delta,\mathcal{B})$.

Proof. Let $u(t,x_1,x_2) := \{\Phi_\varrho^*(e^{-t\Delta_\mathcal{B}}\phi)\}(x_1,x_2)$. By naturality, u satisfies the relations of (1.1.e). As the static operator determined by $\mathcal{D}_\varrho$ is $\Delta+$ lower order terms, $d\nu_M$ is independent of ϱ. Thus

$$K(t,x_1,x_2,\bar{x}_1,\bar{x}_2,\mathcal{D}_\varrho,\mathcal{B}_\varrho) = K(t,x_1+\varrho t\Xi(x_1,x_2),x_2,\bar{x}_1,\bar{x}_2,\Delta,\mathcal{B}).$$

We set $x_1 = \bar{x}_1$ and $x_2 = \bar{x}_2$. We work modulo terms which are $O(\varrho^2)$ and expand in a Taylor series to compute:

$$\begin{aligned}a(f,\mathcal{D}_\varrho,\mathcal{B}_\varrho)(t) &= \textstyle\int_M f(x_1,x_2)K(t,x_1,x_2,x_1,x_2,\mathcal{D}_\varrho,\mathcal{B}_\varrho)d\nu_M\\ &= \textstyle\int_M f(x_1,x_2)K(t,x_1+\varrho t\Xi,x_2,x_1,x_2,\Delta,\mathcal{B})gdx_1dx_2\\ &\equiv \textstyle\int_M\{f(x_1,x_2)K(t,x_1,x_2,x_1,x_2,\Delta,\mathcal{B})\\ &\quad + t\varrho f\Xi\partial_1K(t,x_1,x_2,y_1,x_2,\Delta,\mathcal{B})|_{x_1=y_1}\}gdx_1dx_2.\end{aligned}$$

As $\Delta_\mathcal{B}$ is self adjoint, the heat kernel is symmetric. Thus we have:

$$\begin{aligned}a(f,\mathcal{D}_\varrho,\mathcal{B}_\varrho)(t) &\equiv \textstyle\int_M\{f(x_1,x_2)K(t,x_1,x_2,x_1,x_2,\Delta,\mathcal{B})\\ &\quad + \tfrac{1}{2}t\varrho f\Xi\partial_1K(t,x_1,x_2,x_1,x_2,\Delta,\mathcal{B})\}gdx_1dx_2\\ &\equiv \textstyle\int_M\{f(x_1,x_2)K(t,x_1,x_2,x_1,x_2,\Delta,\mathcal{B})\\ &\quad - \tfrac{1}{2}t\varrho g^{-1}\partial_1(gf\Xi)K(t,x_1,x_2,x_1,x_2,\Delta,\mathcal{B})\}d\nu_M\\ &\equiv a(f,\Delta,\mathcal{B})(t) - \tfrac{1}{2}t\varrho a(g^{-1}\partial_1(gf\Xi),\Delta,\mathcal{B})(t). \quad \square\end{aligned}$$

We use Lemma 5.1 to complete the proof of Theorem 1.3 by completing the calculation of the coefficients c_i and $e_i^\pm$.

5.2 Lemma.

(1) $c_9 = 15$ *and* $c_{10} = -30$.

(2) $e_4^- = -60$, $e_5^- = 30$, $e_6^- = 30$, $e_7^- = -30$, *and* $e_8^- = 0$.

(3) $e_4^+ = 120$, $e_5^+ = -150$, $e_6^+ = -60$, $e_7^+ = 60$, $e_8^+ = 0$, $e_9^+ = 150$, $e_{14}^+ = 360$, *and* $e_{15}^+ = 0$.

Proof. We introduce an auxiliary parameter ε and work modulo terms which are $O(\varepsilon^2)+O(\varrho^2)$. Let

$$ds^2 := e^{2\varepsilon\psi_1}dx_1^2 + e^{2\varepsilon\psi_2}dx_2^2.$$

The Laplacian $\Delta = -g^{-1}\partial_i g g^{ij}\partial_j$ can then be expressed in the form

$$\Delta \equiv -\{e^{-2\varepsilon\psi_1}\partial_1^2 + e^{-2\varepsilon\psi_2}\partial_2^2 + \varepsilon(\psi_{2/1}-\psi_{1/1})\partial_1 + \varepsilon(\psi_{1/2}-\psi_{2/2})\partial_2\}.$$

Let $\Phi_\varrho(t,x_1,x_2) = (t, x_1+\varrho t\Xi, x_2)$. Let $\Xi_{/i} = \partial_i\Xi$ etc. As Φ_ϱ is a diffeomorphism, we can pull back both differential forms and differential operators. We compute:

$$\Phi_\varrho^*(\partial_1) \equiv \partial_1 - t\varrho\Xi_{/1}\partial_1,\ \Phi_\varrho^*(\partial_2) \equiv \partial_2 - t\varrho\Xi_{/2}\partial_1, \Phi_\varrho^*(\partial_t) \equiv \partial_t - \varrho\Xi\partial_1.$$

The operator $\mathcal{D}_\varrho := \Phi_\varrho^*(\partial_t+\Delta) - \partial_t$ is given by:

$$\begin{aligned}\mathcal{D}_\varrho \equiv& \Delta + t\varrho\{e^{-2\epsilon\psi_1}[2\Xi_{/1}\partial_1^2 + \Xi_{/11}\partial_1] + e^{-2\epsilon\psi_2}[2\Xi_{/2}\partial_1\partial_2 + \Xi_{/22}\partial_1]\}\\ &+ t\varrho\varepsilon\{2\psi_{1/1}\Xi\partial_1^2 + 2\psi_{2/1}\Xi\partial_2^2 + \Xi_{/1}(\psi_{2/1}-\psi_{1/1})\partial_1\\ &- \Xi(\psi_{2/11}-\psi_{1/11})\partial_1 + \Xi_{/2}(\psi_{1/2}-\psi_{2/2})\partial_1\\ &- \Xi(\psi_{1/12}-\psi_{2/12})\partial_2\}.\end{aligned}$$

The tensors E, $\mathcal{G}$, and $\mathcal{E}_1$ are therefore given by:

$\mathcal{D}_0 = \Delta - \varrho\Xi\partial_1$	$\omega_1^{\mathcal{D}} \equiv \frac{1}{2}e^{2\varepsilon\psi_1}\varrho\Xi$
$\mathcal{G}_{1,}{}^{11} \equiv e^{-2\varepsilon\psi_1}2\varrho\Xi_{/1} + 2\varepsilon\psi_{1/1}\varrho\Xi$	$\omega_2^{\mathcal{D}} \equiv 0$
$\mathcal{G}_{1,}{}^{22} \equiv 2\varepsilon\psi_{2/1}\varrho\Xi$	$\mathcal{G}_{1,}{}^{12} \equiv e^{-2\varepsilon\psi_2}\varrho\Xi_{/2}$
$E \equiv -\frac{1}{2}\varrho\Xi_{/1} - \frac{1}{2}\varepsilon(\psi_{1/1}+\psi_{2/1})\varrho\Xi$	$\mathcal{E}_1 \equiv 0$

To compute $\mathcal{F}$, we must express partial differentiation in terms of covariant differentiation. Since ω is linear in ϱ, it plays no role. The Christoffel symbols of the metric, however, play a crucial role. We compute:

$$\begin{aligned}&\mathcal{G}_{1,}{}^{11}f_{;11} \equiv (\mathcal{G}_{1,}{}^{11}\partial_1^2 - 2\varrho\Xi_{/1}\varepsilon\psi_{1/1}\partial_1 + 2\varrho\Xi_{/1}\varepsilon\psi_{1/2}\partial_2)f\\ &2\mathcal{G}_{1,}{}^{12}f_{;12} \equiv (2\mathcal{G}_{1,}{}^{12}\partial_1\partial_2 - 2\varrho\Xi_{/2}\varepsilon\psi_{1/2}\partial_1 - 2\varrho\Xi_{/2}\varepsilon\psi_{2/1}\partial_2)f\\ &\mathcal{G}_{1,}{}^{22}f_{;22} \equiv \mathcal{G}_{1,}{}^{22}\partial_2^2 f\end{aligned}$$

We use this computation to determine the tensor $\mathcal{F}_1$:

$$\begin{aligned}\mathcal{F}_{1,}{}^1 \equiv& \varrho(e^{-2\varepsilon\psi_1}\Xi_{/11} + e^{-2\varepsilon\psi_2}\Xi_{/22})\\ &+\varepsilon\varrho\{(\psi_{2/1}-\psi_{1/1})\Xi_{/1} - (\psi_{2/11}-\psi_{1/11})\Xi\\ &+(\psi_{1/2}-\psi_{2/2})\Xi_{/2} + 2\psi_{1/1}\Xi_{/1} + 2\psi_{1/2}\Xi_{/2}\}\\ \mathcal{F}_{1,}{}^2 \equiv& \varepsilon\varrho\{-(\psi_{1/12}-\psi_{2/12})\Xi - 2\psi_{1/2}\Xi_{/1} + 2\psi_{2/1}\Xi_{/2}\}\end{aligned}$$

We now prove assertion (1). Let $P \in \text{int}(M)$. Let $\varepsilon\psi_1(P) = \varepsilon\psi_2(P) = 0$. We study monomials $\Xi_{/111}$ and $\psi_{2/111}\Xi$ appearing in $\frac{\partial}{\partial\varrho}\{a_4^M(\cdot)\}|_{\varrho=0}$. Let $\mathcal{R} = E$ or let $\mathcal{R} = R_{ijji}$. We integrate by parts to define $\mathcal{A}[\mathcal{R}]$ by the identity:

$$\begin{aligned}&-\tfrac{1}{12}\textstyle\int_M g^{-1}\partial_1(gf\Xi)\mathcal{R}d\nu_M = \tfrac{1}{360}\int_M f\mathcal{A}[\mathcal{R}]d\nu_M;\ \text{then}\\ &-\tfrac{1}{2}a_2^M(g^{-1}\partial_1(gf\Xi),\Delta) = (4\pi)^{-1}\tfrac{1}{360}\textstyle\int_M f\mathcal{A}[6E+R_{ijji}]d\nu_M.\end{aligned}$$

We have $R_{ijji} \equiv -2\varepsilon\psi_{2/11} + ...$ We compute:

$$\begin{array}{lcrr}
\frac{\partial}{\partial\varrho}\{60E_{;ii}\}|_{\varrho=0} & \equiv & -30\Xi_{/111} & -30\varepsilon\psi_{2/111}\Xi + ...\\
\frac{\partial}{\partial\varrho}\{60\mathcal{F}_{1,i;i}\}|_{\varrho=0} & \equiv & 60\Xi_{/111} & -60\varepsilon\psi_{2/111}\Xi + ...\\
\frac{\partial}{\partial\varrho}\{c_9\mathcal{G}_{1,ii;jj}\}|_{\varrho=0} & \equiv & 2c_9\Xi_{/111} & +2c_9\varepsilon\psi_{2/111}\Xi + ...\\
\frac{\partial}{\partial\varrho}\{c_{10}\mathcal{G}_{1,ij;ij}\}|_{\varrho=0} & \equiv & 2c_{10}\Xi_{/111} & +0c_{10}\varepsilon\psi_{2/111}\Xi + ...\\
\mathcal{A}[6E] & \equiv & 0\Xi_{/111} & +0\varepsilon\psi_{2/111}\Xi + ...\\
\mathcal{A}[R_{ijji}] & \equiv & 0\Xi_{/111} & -60\varepsilon\psi_{2/111}\Xi + ...
\end{array}$$

We use Lemma 5.1 to relate the coefficients of $f\Xi_{/111}$ and $f\psi_{2/111}\Xi$ and establish the following relationships from which assertion (1) follows:

$$-30 + 60 + 2c_9 + 2c_{10} = 0 \text{ and } -30 - 60 + 2c_9 = -60.$$

We now study the boundary terms. We pullback the Robin boundary operator

$$\Phi_\varrho^*(e^{-\varepsilon\psi_2}\partial_2 + S) \equiv e^{-\varepsilon\psi_{2/1}t\varrho\Xi}\{\mathcal{B} - e^{-\varepsilon\psi_2}t\varrho\Xi_{/2}\partial_1 + t\varrho\Xi(S\varepsilon\psi_{2/1} + S_{/1})\}$$

to determine the tensors

$$T^1 \equiv -e^{-\varepsilon\psi_2}\varrho\Xi_{/2} \text{ and } S_1 \equiv \varrho\Xi(\varepsilon\psi_{2/1}S + S_{/1}).$$

We have $L_{11} \equiv -\varepsilon\psi_{1/2}$. We study the terms comprising $\frac{\partial}{\partial\varrho}\{a_4^{\partial M}(f,\mathcal{D}_\varrho,\mathcal{B}_\varrho)\}|_{\varrho=0}$. At the point of the boundary in question, we suppose $\varepsilon\psi_1(P) = \varepsilon\psi_2(P) = 0$.

$$\begin{aligned}
&\tfrac{\partial}{\partial\varrho}\{(-120^-,240^+)fE_{;m}\}|_{\varrho=0}\\
&\qquad\equiv (60^-,-120^+)f\{\Xi_{/12} + (\varepsilon\psi_{1/12} + \varepsilon\psi_{2/12})\Xi + (\varepsilon\psi_{1/1} + \varepsilon\psi_{2/1})\Xi_{/2}\},\\
&\tfrac{\partial}{\partial\varrho}\{120fEL_{aa}\}|_{\varrho=0} \equiv 60\varepsilon f\psi_{1/2}\Xi_{/1},\\
&\tfrac{\partial}{\partial\varrho}\{720fSE\}|_{\varrho=0} \equiv -360fS\{\Xi_1 + \varepsilon(\psi_{1/1} + \psi_{2/1})\Xi\},\\
&\tfrac{\partial}{\partial\varrho}\{e_3^\pm f\mathcal{G}_{1,aa}L_{bb}\}|_{\varrho=0} \equiv e_3^\pm f(2\Xi_{/1})(-\varepsilon\psi_{1/2}),\\
&\tfrac{\partial}{\partial\varrho}\{e_4^\pm f\mathcal{G}_{1,mm}L_{bb}\}|_{\varrho=0} \equiv 0,\\
&\tfrac{\partial}{\partial\varrho}\{e_5^\pm f\mathcal{G}_{1,ab}L_{ab}\}|_{\varrho=0} \equiv e_5^\pm f(2\Xi_{/1})(-\varepsilon\psi_{1/2}),\\
&\tfrac{\partial}{\partial\varrho}\{e_6^\pm f\mathcal{G}_{1,mm;m}\}|_{\varrho=0} \equiv e_6^\pm f(2\varepsilon\psi_{2/12}\Xi + 4\varepsilon\psi_{2/1}\Xi_{/2}),\\
&\tfrac{\partial}{\partial\varrho}\{e_7^\pm f\mathcal{G}_{1,aa;m}\}|_{\varrho=0} \equiv e_7^\pm f\{2\Xi_{/12} + 2\varepsilon\psi_{1/12}\Xi + 2\varepsilon\psi_{1/1}\Xi_{/2} - 2\varepsilon\psi_{2/1}\Xi_{/2}\},\\
&\tfrac{\partial}{\partial\varrho}\{e_8^\pm f\mathcal{G}_{1,am;a}\}|_{\varrho=0} \equiv e_8^\pm f\{-\varepsilon\psi_{2/1}\Xi_{/2} + \Xi_{/12} + \varepsilon\psi_{1/1}\Xi_{/2} - 2\varepsilon\psi_{1/2}\Xi_{/1}\},
\end{aligned}$$

$\frac{\partial}{\partial\varrho}\{e_9^\pm f\mathcal{F}_{1,m}\}|_{\varrho=0} \equiv e_9^\pm f\{-(\varepsilon\psi_{1/12}-\varepsilon\psi_{2/12})\Xi - 2\varepsilon\psi_{1/2}\Xi_{/1} + 2\varepsilon\psi_{2/1}\Xi_{/2}\}$,
$\frac{\partial}{\partial\varrho}\{e_{12}^+ fS\mathcal{G}_{1,aa}\}|_{\varrho=0} \equiv e_{12}^+ f\{2\Xi_{/1}S + 2\varepsilon\psi_{1/1}\Xi S\}$,
$\frac{\partial}{\partial\varrho}\{e_{13}^+ fS\mathcal{G}_{1,mm}\}|_{\varrho=0} \equiv e_{13}^+ f\{2\varepsilon\psi_{2/1}\Xi S\}$,
$\frac{\partial}{\partial\varrho}\{e_{14}^+ fS_1\}|_{\varrho=0} \equiv e_{14}^+ f\Xi\{\varepsilon\psi_{2/1}S + S_{/1}\}$,
$\frac{\partial}{\partial\varrho}\{e_{15}^+ fT_{a:a}\}|_{\varrho=0} \equiv e_{15}^+ f(\varepsilon\psi_{2/1}\Xi_{/2} - \Xi_{/12} - \varepsilon\psi_{1/1}\Xi_{/2})$,
$\frac{\partial}{\partial\varrho}\{(\pm 180)f_{;m}E\}|_{\varrho=0} \equiv \mp 90 f_{;m}\{\Xi_{/1} + (\varepsilon\psi_{1/1}+\varepsilon\psi_{2/1})\Xi\}$,
$\frac{\partial}{\partial\varrho}\{e_{10}^\pm f_{;m}\mathcal{G}_{1,aa}\}|_{\varrho=0} \equiv e_{10}^\pm f_{;m}(2\Xi_{/1} + 2\varepsilon\psi_{1/1}\Xi)$,
$\frac{\partial}{\partial\varrho}\{e_{11}^\pm f_{;m}\mathcal{G}_{1,mm}\}|_{\varrho=0} \equiv e_{11}^\pm f_{;m}2\varepsilon\psi_{2/1}\Xi$.

We must also study the boundary terms comprising $-\frac{1}{2}a_2^{\partial M}(\cdot)$. As when studying a_2^M, we integrate by parts to define $\mathcal{A}$ and compute:

$\mathcal{A}[2fL_{aa}] \equiv -60\varepsilon f\psi_{1/12}\Xi$,
$\mathcal{A}[12fS] \equiv -360\{\Xi\varepsilon fS\psi_{2/1} - f\Xi S_{/1}\}$,
$\mathcal{A}[\pm 3f_{;m}] \equiv \mp 90\{(\varepsilon\psi_{1/12}+\varepsilon\psi_{2/12})f\Xi + 2\varepsilon\psi_{2/1}(f_{;m}\Xi + f\Xi_{/2})\}$.

We established the following relations in Lemmas 3.2 and 4.2:

$$e_3^\pm = 30, e_4^\pm + e_5^\pm = -30,\ e_{14}^+ - 2e_9^+ = 60 \text{ and } e_9^- = -30.$$

We use Lemma 5.1 to derive the following equations and complete the proof:

$(60^-, -120^+) + 4e_6^\pm - 2e_7^\pm - e_8^\pm + 2e_9^\pm + e_{15}^\pm = \mp 180$	$[f\varepsilon\psi_{2/1}\Xi_{/2}]$		
$(60^-, -120^+) + 2e_6^\pm + e_9^\pm = \mp 90$	$[f\varepsilon\psi_{2/12}\Xi]$		
$(60^-, -120^+) + 2e_7^\pm - e_9^\pm = -60 \mp 90$	$[f\varepsilon\psi_{1/12}\Xi]$		
$(60^-, -120^+) + 2e_7^\pm + e_8^\pm - e_{15}^\pm = 0$	$[f\Xi_{/12}]$		
$-2e_5^\pm - 2e_8^\pm - 2e_9^\pm = 0$	$[f\varepsilon\psi_{1/2}\Xi_{/1}]$	$e_{14}^+ = 360$	$[fS_{/1}\Xi]$
$-360 + 2e_{13}^+ + e_{14}^+ = -360$	$[f\varepsilon\psi_{2/1}\Xi S]$	$-360 + 2e_{12}^+ = 0$	$[f\Xi_{/1}S]$
$\mp 90 + 2e_{11}^\pm = \mp 180$	$[f_{;m}\varepsilon\psi_{2/1}\Xi]$	$\mp 90 + 2e_{10}^\pm = 0$	$[f_{;m}\Xi_{/1}]$

§6 Commuting operators

We conclude this paper by deriving a final functorial property. The equations which can be derived using this property are compatible with the values for the constants c_i and $e_i^\pm$ previously computed; they are omitted in the interests of brevity.

6.1 Lemma. *Let D be a self-adjoint static operator of Laplace type and let $\mathcal{B}$ be a static boundary condition. Let Q be an auxiliary self-adjoint static partial differential operator of order at most 2 which commutes with D and with $\mathcal{B}$. Then:*

$$\frac{\partial}{\partial\varrho}\{a_n(f, D + 2t\varrho Q, \mathcal{B})\}|_{\varrho=0} = \frac{\partial}{\partial\varrho}\{a_{n-2}(f, D + \varrho Q, \mathcal{B})\}|_{\varrho=0}.$$

Remark. If we take $D = Q$, then $D(\varrho) = (1 + 2t\varrho)D$. By Lemma 3.1,

$$\tfrac{\partial}{\partial\varrho}\{a_4(f,(1+2t\varrho)D,\mathcal{B})\}|_{\varrho=0} = \tfrac{2-m}{2}a_2(f,D,\mathcal{B}).$$

On the other hand, clearly $a_n(f,(1+\varrho)D,\mathcal{B}) = (1+\varrho)^{(n-m)/2}a_n(f,D,\mathcal{B})$. Thus we may show that Lemma 6.1 is compatible with Lemma 3.1 in this special case by computing:

$$\tfrac{\partial}{\partial\varrho}\{a_2(f,(1+\varrho)D,\mathcal{B})\}|_{\varrho=0} = \tfrac{2-m}{2}a_2(f,D,\mathcal{B}) = \tfrac{\partial}{\partial\varrho}\{a_4(f,D+2t\varrho D,\mathcal{B})\}|_{\varrho=0}.$$

Proof. Let $\mathcal{K}_1(t) := (1-t^2\varrho Q)e^{-tD_\mathcal{B}}$. Then $\mathcal{K}_1(0)$ is the identity operator and:

$$\begin{aligned}&(\partial_t + D + 2t\varrho Q)(1-t^2\varrho Q)e^{-tD_\mathcal{B}}\\ =&\{-2t\varrho Q - (1-t^2\varrho Q)D + D(1-t^2\varrho Q) + 2t\varrho Q(1-t^2\varrho Q)\}e^{-tD_\mathcal{B}}\\ =&-2t^3\varrho^2Q^2e^{-tD_\mathcal{B}}.\end{aligned}$$

There exists a constant C and an integer μ so that we have the estimate in a suitable operator norm:

$$|-2t^3\varrho^2Q^2e^{-tD_\mathcal{B}}| \le Ct^{-\mu}\varrho^2.$$

Thus since we are interested in the linear terms in ϱ, we may replace the fundamental solution of the heat equation $\mathcal{K}(t)$ for $D + 2t\varrho Q$ by the approximation $(1-\varrho t^2Q)e^{-tD_\mathcal{B}}$. There is an asymptotic expansion of the form [4]:

$$\mathrm{Tr}_{L^2}(fQe^{-tD_\mathcal{B}}) \sim \textstyle\sum_{n\ge0} t^{(n-m-2)/2}a_n(f,Q,D,\mathcal{B}).$$

We equate coefficients of $t^{(n-m)/2}$ in the asymptotic expansions to see

$$\tfrac{\partial}{\partial\varrho}\{a_n(f,D+2t\varrho Q,\mathcal{B})\}|_{\varrho=0} = -a_{n-2}(f,Q,D,\mathcal{B}).$$

Since Q and D commute and since Q and $\mathcal{B}$ commute, we complete the proof by computing:

$$\begin{aligned}&\textstyle\sum_{n\ge0}\tfrac{\partial}{\partial\varrho}\{a_n(f,D+\varrho Q,\mathcal{B})\}|_{\varrho=0}t^{(n-m)/2} \sim \tfrac{\partial}{\partial\varrho}\{\mathrm{Tr}_{L^2}(fe^{-t((D+\varrho Q)_\mathcal{B})})\}|_{\varrho=0}\\ &\quad = \mathrm{Tr}_{L^2}(-tfQe^{-tD_\mathcal{B}}) \sim -\textstyle\sum_{n\ge0}a_n(f,Q,D,\mathcal{B})t^{(n-m)/2} \text{ so}\\ &\tfrac{\partial}{\partial\varrho}\{a_n(f,D+\varrho Q,\mathcal{B})\}|_{\varrho=0} = -a_n(f,Q,D,\mathcal{B}). \quad \square\end{aligned}$$

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References

[1] N. D. Birrell and P. C. W. Davies, *Quantum fields in Curved Space*, Cambridge University Press, Cambridge, 1982.

[2] T. Branson and P. Gilkey, *The Asymptotics of the Laplacian on a manifold with boundary*, Comm. in PDE **15(2)** (1990), 245–272.

[3] T. Branson, P. Gilkey, K. Kirsten, and D. Vassilevich, *Heat kernel asymptotics with mixed boundary conditions*, Nuclear Physics B **563** (1999), 603–626.

[4] P. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index theorem (2^{nd} edition)*, CRC Press, Boca Raton, Florida, ISBN 0-8493-7874-4 (1994).

[5] ———, *The heat content asymptotics for variable geometries*, J. Phys: A: Math. Gen. **32** (1999), 2825–2834.

[6] G. Grubb, *Functional calculus of pseudo differential boundary problems*, Progress in Math. 65, Birkhäuser, Boston, 1986.

[7] G. Kennedy, R. Critchley, and J. S. Dowker, *Finite Temperature Field Theory with Boundaries: Stress Tensor and Surface Action Renormalization*, Annals of Physics **125** (1980), 346–400.

[8] I. Moss, *Boundary terms in the heat kernel expansion*, Class. Quantum Grav. **6** (1989), 759–765.

[9] I. Moss and J. S. Dowker, *The Correct B_4 Coefficient*, Phys. Letts. B. **229** (1989), 261–263.

[10] H. Weyl, *The Classical Groups*, Princeton Univ. Press, Princeton, 1946.

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